Event analysis of β decay

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Normally, the β-decay half-life of a nuclide is determined by measuring the number of decay events in a series of consecutive time intervals (channels) and presenting the results as a function of elapsed time in the form of a histogram known as the decay spectrum. In the *histogram analysis* that follows, the half-life is evaluated based on the known time-dependence of the event rate expected under ideal conditions (i.e., in the absence of the detection system's dead time), as described by a function that will be denoted by ρ and referred to as the *ideal rate function*. For example, in the case of a single-component decay in the presence of a constant background B,

$$\rho = A \exp(-\lambda t) + B , \qquad (1)$$

where A is the initial ideal rate (at time $t = 0$) in the absence of background, and

$$\lambda = \ln(2) / T_{1/2} \qquad (2)$$

is the nuclide-specific decay constant, which is related to the nuclide's half-life $T_{1/2}$. In this report it is assumed that the system's detection efficiency does not depend on ρ. Specifically, in the *maximum-likelihood* method of histogram analysis, the best estimates of the parameters A, $T_{1/2}$, and B are taken to be those that maximize the resulting probability of obtaining the actual β-decay spectrum as a result of a measurement.

In high-precision work, there are three major disadvantages of histogram analysis. The first one is related to the fact that ρ changes as a function of time within each channel. Although this effect can be easily accounted for under ideal conditions, it cannot be otherwise: the statistics of actual events are distorted by the unavoidable presence of the detection system's dead time. All theoretical results that are relevant and apply under these circumstances have been derived assuming that ρ is constant. While the range of ρ values within a channel can be limited by decreasing the channel width, ρ varies by about 3 % per channel in typical measurements (for various technical reasons).

The second major disadvantage of histogram analysis is related to the fact that the probability (on which the maximum-likelihood method is based) of obtaining the measured number of events in a given channel, can be calculated exactly (based on ρ) only under ideal conditions. In the presence of a fixed non-extendable dead time this is possible only if ρ is constant. The theoretical results for other conditions (including the case of a fixed extendable dead time) are not available. Furthermore, the actual dead time is not likely to be fixed and non-extendable, and it is not even likely to be accurately predictable [1]. In a typical attempt to work around this problem, a known fixed non-extendable dead time is inserted in the detection system (preferably at an early stage) before the measurement starts. This dead time is set to be dominant, so that other contributions to the detection system's dead time are expected to become negligible. Consequently, only the dominant dead time is considered in the analysis. The remaining problem is that the dominant dead time should be set to a minimum effective value, which is hard to predict. Therefore, the measurements must be repeated with different values of the dominant dead time in order to examine their effect on the results. However, even if it is found that an increased dominant dead time does not change the best estimates of the parameters of ρ (within their statistical uncertainties), that does not imply that the detection system's dead time can be accurately described as being fixed and non-extendable, and for high counting rates does not justify basing the analysis on this assumption. One way of mitigating this deficiency is to reduce the β-decay rate enough to make the uncertainties of the dead-time corrections negligible. However, in that case, the number of events in the spectrum is reduced, so that the measurements have to be repeated in order to reduce the uncertainty of the results, which increases the time it takes to complete the measurements and the analysis.

The third major disadvantage of histogram analysis is related to the required numerical accuracy and the desired speed of the calculations involved. Both issues become critical when the dead-time effect is significant and the number of counts in a single channel is relatively high (more than 1,000). This problem can be reduced by decreasing the channel width and/or by decreasing the β-decay rate.

With the TDC-based data acquisition system [1], the arrival time of each β-decay event is measured individually, and so the β-decay spectra can be constructed on-line or off-line, using any chosen value for the channel width. This chosen value can be set as low as needed in order to sufficiently reduce the effect of rate change within each channel and the numerical error in the calculations. Likewise, the effect that the dominant dead time would have if inserted at the final stage of event-detection can be simulated by software, which eliminates the need to insert the dominant dead time by hardware as well as the need for repeated measurements at different dominant dead times.

Ultimately, taking advantage of the available individual event timing information provided by the TDC-based system [1], the data can be analyzed event by event, without the need to construct the β-decay spectra (except for the sake of visualization). This report describes the principles behind such an analysis as well as some of the results. The method proposed here will be referred to as *event analysis*.

To begin with, let us assume that ρ is constant and that the conditions are ideal. In this case, based on Poisson statistics, the probability dp that an event occurring at time zero will be followed by the next event in the time interval $[t\ ,\ t + \mathrm{d}t)$ is given by

$$\mathrm{d}p = P_0\ \rho\ \mathrm{d}t\ , \tag{3}$$

where

$$P_0 = \exp(-\rho t) \tag{4}$$

is the probability that no events occur in the time interval $[0,\ t)$, while $\rho\ \mathrm{d}t$ is the probability that an event occurs in the time interval $[t,\ t + \mathrm{d}t)$. Therefore,

$$\mathrm{d}p = \exp(-\rho t)\ \rho\ \mathrm{d}t\ . \tag{5}$$

If ρ is not constant, then eq.(4) must be generalized, so that, exactly,

$$P_0 = \exp(- \langle\rho\rangle\, t)\ , \tag{6}$$

where

$$\langle \rho \rangle = \frac{1}{t}\int_0^t \rho dt \tag{7}$$

is the average value of ρ over the time interval $[0, t)$. Accordingly, eq.(5) becomes

$$\mathrm{d}p = \exp(- \langle\rho\rangle\, t)\, \rho_t\, \mathrm{d}t\ , \tag{8}$$

where ρ_t is the value of ρ at time t.

In the presence of a known detection system's dead time per pulse τ, eq.(8) becomes (exactly)

$$\mathrm{d}p = \Theta(t - \tau)\, \exp[- \langle\rho\rangle_\tau\, (t - \tau)]\, \rho_t\, \mathrm{d}t\ , \tag{9}$$

where

$$\langle \rho \rangle_\tau = \frac{1}{t-\tau}\int_\tau^t \rho dt\ \ , \tag{10}$$

and $\Theta(t - \tau)$ is the Heaviside (unit-step) function.

The goal of data analysis is to determine the best estimates and uncertainties of the parameters of ρ. In event analysis, this is accomplished by choosing time zero at the time of detection of the first β-decay event and calculating, for each event i, the time $\Delta t_i = t_i - t_{i-1}$ elapsed since the previous event (i-1). Here i ranges from 2 to the index N of the last event in the current measurement cycle. This is followed by calculating the quantity W, which is proportional to the probability that the entire set of time differences Δt_i ($i = 2, \ldots, N$) is measured as it was, using

$$W = \prod_{i=2}^{N} \rho_{ti} \exp[- \langle \rho_i \rangle_{\tau i}\, (\Delta t_i - \tau_i)]\ . \tag{11}$$

Note that the Heaviside functions were left out because, for real events and correctly determined dead times, their values must always equal 1. Finally, the parameters of ρ are varied iteratively in order to minimize the quantity E, defined as the natural logarithm of W multiplied by -2, i.e.,

$$E = -2\sum_{i=2}^{N} [\ln(\rho_{ti}) - \langle \rho_i \rangle_{\tau i}\, (\Delta t_i - \tau_i)]\ . \tag{12}$$

In each iteration, the value of ρ can be calculated for any instant and its average value can be calculated for any time interval based on the current values of its parameters. The error of any parameter of ρ can be calculated as the square root of the corresponding diagonal element of the inverse of the Hessian matrix of E.

For a fixed non-extendable dead time per pulse, τ does not depend on i, so its index can be dropped, while for the case of a fixed extendable dead time, eq.(9) must be replaced by the corresponding expression that applies to that case and eqs.(11-12) must be modified accordingly.

One way of assessing the quality of the results is to construct the histograms (β-decay spectra) of the measured events, compare them with the predictions based on the best estimate of ρ, and then proceed using the methods of histogram analysis. This has been done in order to compare the results of the event analysis with those of the histogram analysis for an extensive set of simulated ideal data on which a fixed non-extendable dead time was imposed. It was found that the two sets of results are identical for all practical purposes. However, the event analysis was found to be faster by a factor of 6. Also, using simulated data, the event analysis was confirmed to produce the expected results at high counting rates (more than 100,000 per second), and in the case of fixed extendable dead time.

In addition, the data collected using the TDC-based system and subject to the event analysis offer

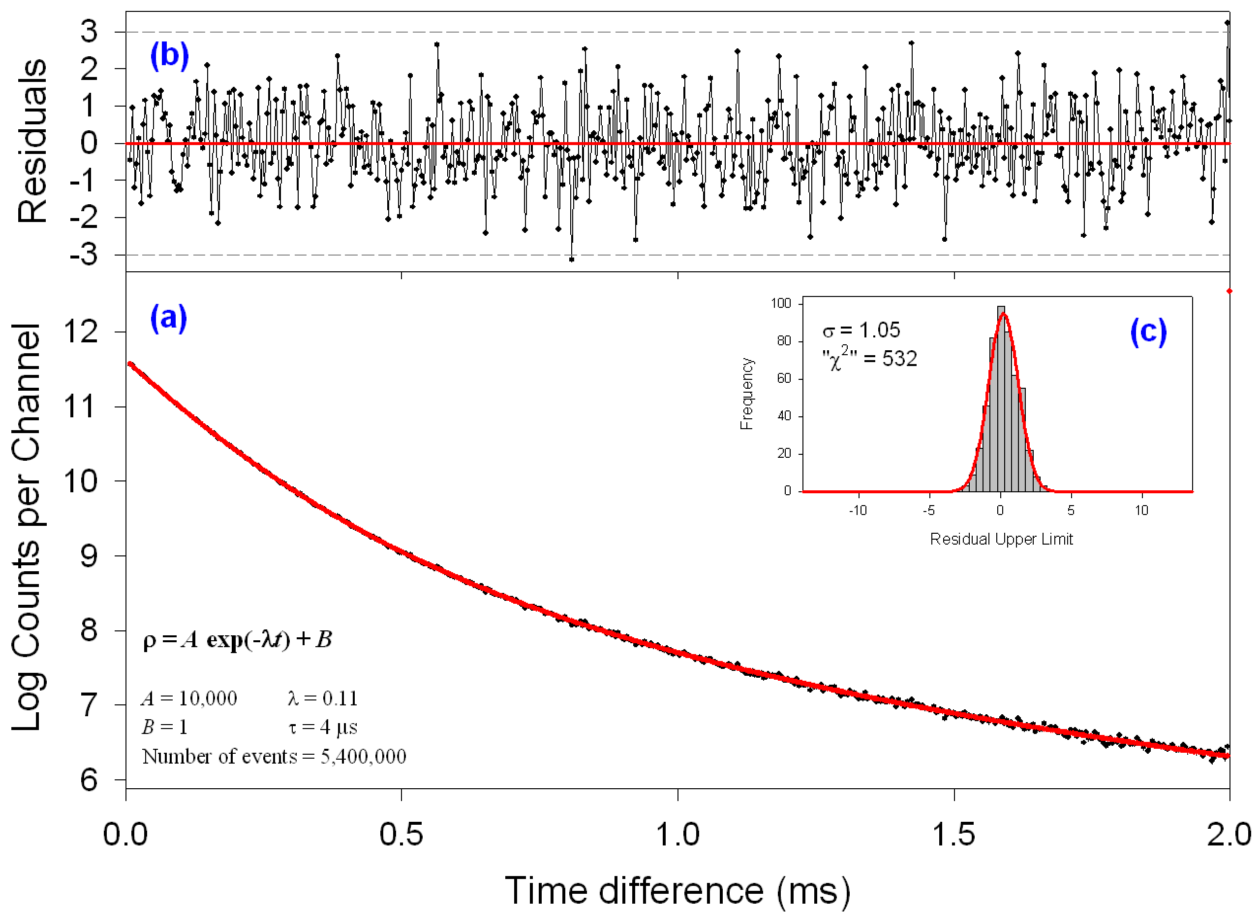

FIG. 1. (a) Time-difference spectrum (shown by black circles) of simulated ideal data to which a fixed non-extendable dead time of 4 μs was imposed. The red circles represent the values predicted based on ρ (as defined in the text). The last data point includes the contribution for all the events with time-difference exceeding 2 ms. (b) Spectrum of the residuals. (c) Histogram of the distribution of the residuals (shown by grey bars). The red line represents the best-fit Gaussian curve.

another method of result-quality assessment. Namely, the measured events can be used to construct a histogram of Δt values (i.e., the Δt spectrum, or the time-difference spectrum), and the result can be compared with the prediction based on the best estimate of ρ. By looking at the spectrum and the distribution of the residuals, it is possible to assess whether the predicted Δt spectrum provides an accurate description of the measured Δt spectrum for all values of Δt. If this is not the case, the assumed parameterization and time-dependence of ρ, as well as the assumptions regarding the nature and extent of the dead time must be questioned. As an example, results of the event analysis of a set of simulated ideal data with a fixed non-extendable dead time imposed on them are shown in Figure 1. Here it should be noted that the measured Δt spectrum was not fitted in the analysis. Instead, the predicted Δt spectrum was calculated based on ρ, where the best estimates of its parameters were obtained by minimizing the function E of eq.(12).

[1] V. Horvat and J.C. Hardy, *Progress in Research*, Cyclotron Institute, Texas A&M University (2010-2011), p. V-51.